Electric field of continuous charge distribution

* As we have learned that charge is discrete, or quantized; that is, it always occurs in integer multiples of the fundamental charge 'e'.
However the charge distribution of macroscopic bodies can be treated as continuous fluid with a charge density spread out over the body.

In General we encounter three types of continuous charge distribution

(1) Linear charge distribution:
Charge distributed on a string, a wire, or a thin rod charge is spread out along the line with charge per unit length λ

then $dq = \lambda dl$

Here dq is small charge in a small linear region of length dl.

The small charge dq contributes a field of magnitude dE at some point, which is 'r' distance away.

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}$$

Here S.I unit of λ is coulomb/meter

Let total charge on line is Q of length L.

then $\lambda = Q/L$

(2) Surface charge distribution

Charge distributed over a flat or curved surface.

charge distributed over the surface with the charge per unit area α, then small charge dq on a small piece of surface area dA ~~is equal to~~ then

$$dq = \alpha \, dA$$

Let Q be the charge on surface, having area A

then $\alpha = Q/A$

S.I unit is C/m^2

The small charge dq contributes a field of magnitude dE at some point which is 'r' distance away

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}$$

(3) Volume charge distribution

charge distributed ~~over~~ through out three dimensional volume.

Here charge distributed per unit volume is ρ.

$\rho = Q/V$ [S.I unit C/m^3]

$\therefore \; dq = \rho \, dV$

$dq \rightarrow$ small charge

$dV \rightarrow$ piece of volume

$$\therefore \; dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}$$

The superposition principle still holds for continuous charge distributions. We need to sum the individual small contributions to the electric field from each small piece of charge. Mathematically, a sum over a continuous region means we integrate the contributions to get the total field. Since the electric field is a vector, we have to perform the integration for each component of dE.

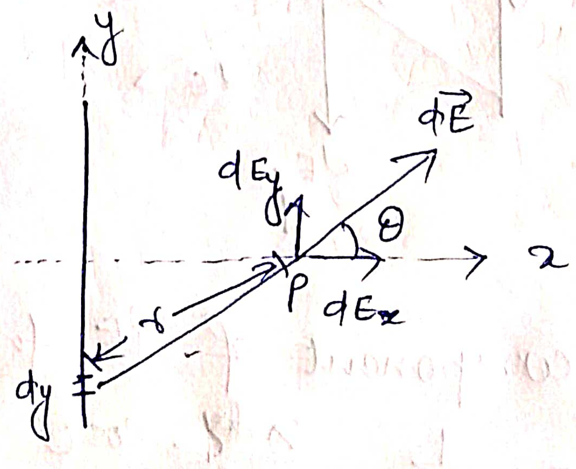
$$\int d\vec{E}(\vec{r}) = \vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r}$$

~~Problem~~

Problem: charge is distributed uniformly along an infinitely long, straight thin rod. If the charge per unit length of the rod is λ, what is the electric field at some distance from the rod?

Solution:



The line element dy carries a charge dq

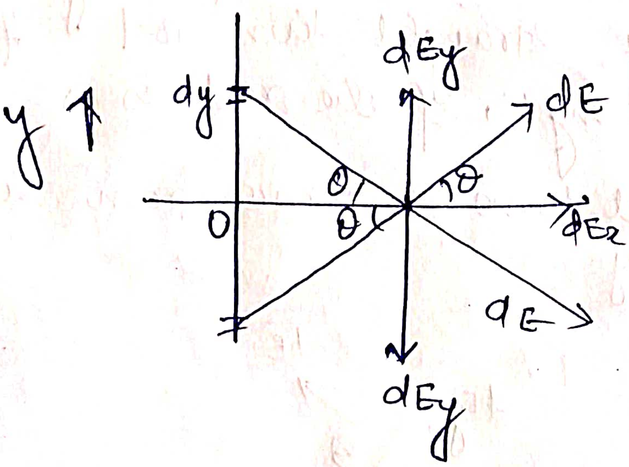
$\Rightarrow dq = \lambda\, dy$

Here the line element can be treated as a point charge, it generates an electric field of magnitude dE.

$$\Rightarrow dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dy}{r^2}$$

- Electric field dE has x and y component

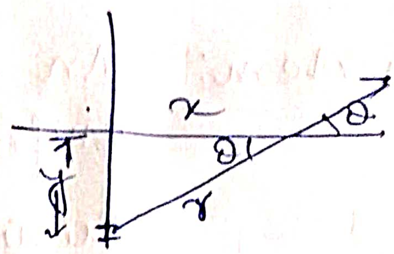
The y component will be cancelled ~~from~~ as upper half will have y component of electric field in negative direction and lower half will have y component of electric field in positive direction. As shown in figure below.



So, the x component of dE for upper half of rod.

$$dE_x = \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dy}{r^2} \cos\theta$$

So to obtain net electric field in x direction we have to integrate over infinitesimal line ~~between~~ elements



$$\therefore \int dE_x = \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{\lambda dy \cos\theta}{r^2}$$

$$\Rightarrow E_x = \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{\lambda dy \cos\theta}{r^2}$$

Now let's solve integration

let $y = x \tan\theta$

$dy = x \sec^2\theta \, d\theta$

$\Rightarrow$ ~~R~~ $\cos\theta = \frac{x}{r}$

$\Rightarrow r = \frac{x}{\cos\theta} \Rightarrow r = x\sec\theta$

On replacing the values in above integral

$$\therefore E_x = \int_{-\pi/2}^{\pi/2} \frac{1}{4\pi\epsilon_0} \frac{\lambda x \sec^2\theta \, d\theta \, \cos\theta}{x^2 \sec^2\theta}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{x} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{x} \left| +\sin\theta \right|_{-\pi/2}^{\pi/2} = \frac{\lambda}{x 4\pi\epsilon_0} [\sin\pi/2 + \sin\pi/2$$

$$= \frac{\lambda}{x} \times \frac{2}{4\pi\epsilon_0} = \frac{\lambda}{x} \frac{1}{2\pi\epsilon_0}$$